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Congestion in production correspondences

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Abstract This contribution aims to detect and measure more severe forms of congestion than the ones that could hitherto be evaluated in axiomatic production theory. To this end, we define a new S-disposal axiom, a kind of limited strong disposability. This S-disposal assumption leads to a duality result between a general input directional distance function and the cost function that is weaker than the ones established in the literature. Finally, we indicate how finite data sets can or cannot be rationalized by a minimal technology compatible with S-disposal, thereby generalizing the nonparametric weak axiom of cost minimization test.

Keywords Distance function · Cost function · Duality · Congestion · WACM

JEL Classification C61 · D24

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1 Introduction

Congestion-intuitively defined as production with negative marginal product- is mainly mentioned as a theoretical curiosity in production theory. For instance, when discussing average and marginal productivity the use of the qualification “irrational” to certain of the so-called “stages of production” points to the low probability attributed to congestion occurring in practice (e.g., Ferguson 1969, 66–79). What seems often ignored is the theory-dependency of observations: to detect a phenomenon, one must have a theoretical framework allowing to observe it. While detailed theoretical studies defining several notions of congestion exist for the single output case (see Färe and Svensson 1980), it is problematic that there are currently no axiomatically founded production technologies to detect all possible congestion phenomena in a multi-output context.

Prominent examples of congestion phenomena are probably traffic congestion and agricultural output loss due to excessive use of fertilizers. Duranton and Turner (2011) offer detailed evidence for a “fundamental law of road congestion” covering a broad class of major US urban roads. Crop response models relating crop yield to nutrients have widely documented limited substitution possibilities, the existence of a maximum yield (plateau) where marginal product of inputs is zero, and even a declining phase of crop yields (see the survey in Paris 2008).[1]

While some forms of congestion appear in economics, this contribution provides an axiomatic foundation to reconstruct technologies capable to reveal “hypercongestion” (loosely described as a total loss of output when inputs are wasted in certain critical combinations). Trivial examples of hypercongestion are the total destruction of a crop due to flooding following a thunderstorm (excess water eventually combined with other climatological circumstances), or traffic jams that temporarily destroy the whole throughput on an arc in a network resulting in a zero traffic flow.

In applied production analysis, many functional forms cannot detect congestion at all. E.g., the Cobb Douglas specification imposes positive marginal productivity along the isoquant throughout input space. Furthermore, the common use of flexible functional forms created a practice of imposing curvature globally, while monotonicity is only imposed locally (to maintain flexibility) or not at all. Barnett (2002) describes and illustrates some of the available evidence indicating that imposing curvature solely can actually induce violations of monotonicity. Without the satisfaction of both curvature and monotonicity, the standard second-order conditions for optimizing behavior fail and duality theory breaks down. Apart from these flexible functional forms, to our knowledge there is only the ray or weakly disposable production function -a generalization of the variable elasticity of substitution function- that can identify congestion. However, it has rarely been applied empirically.[2]

In nonparametric production theory multi-output ray and free disposable technologies have been employed to distinguish between technical inefficiency, understood as production below the production frontier, and congestion, interpreted as a partic-

[1] The latter phase is known as the toxic range of nutrients in soil science.

[2] The few empirical studies using this specification focused mainly on disembodied technical change that widens productive factor combinations rather than detecting congestion.

ular severe form of technical inefficiency. Congestion occurs when either additional inputs actually decrease outputs, or additional input quantities of some input dimension necessitate the opportunity cost of some additional other input dimensions to maintain current output levels (see Färe and Grosskopf 1983). Thus, congestion implies an opportunity cost in either lost output, or some losses in some other input dimensions to maintain current output levels. The empirical analysis of technical efficiency along with productivity has become quite popular recently (see, e.g., Henderson and Russell 2005 revisiting issues on international macroeconomic convergence). However, congestion is often neglected in such studies, despite the fact that some studies indicate it is the most important source of underperformance (e.g., Zhengfei and Oude Lansink 2003).

This contribution focuses upon a new axiom allowing to define more general multi-output technologies capable of revealing the full range of congestion notions defined in Färe and Svensson (1980), including output prohibitive congestion (a technical term denoting hypercongestion). This also requires generalizing their mono-output definitions of congestion for the multi-output context. Since it is important to be able to model all forms of congestion using axiomatically founded technologies, this contribution fills a void in the multi-output literature.

We also look at the implications of these technologies for duality theory. While duality between cost and input distance functions is traditionally established imposing strong disposability of inputs (e.g., Jacobsen 1970; Luenberger 1995), also a weaker duality result between the cost function and the ray (or weakly) disposable input distance function is available whereby some (but not all) prices can be negative (e.g., Shephard 1974). The main purpose of this contribution is to establish a more general duality result based on a limited disposability assumption. This research is driven by a triple motivation.

First, we consider the axiom of ray disposability of inputs intuitively unappealing, since it amounts to assuming that inputs can be disposed off along a ray without any limitation. We suggest to replace this ray disposability assumption with a weaker S-disposal assumption that essentially makes the strong disposability assumption a limited rather than a global property. This is partly inspired by Lau (1974, p. 182) who suggested looking for a local version of strong disposability.[3] In this view, since only variables with values within a certain domain are relevant it suffices to maintain the monotonicity property within a prescribed domain as dictated by sample information. Therefore, the S-disposal assumption can model more general forms of congestion (as defined by Färe and Svensson 1980) than the ray disposability assumption, including the case of limits on the ray disposal of inputs.

Second, given the well-known reasons for nonconvexities in production (e.g., indivisibilities, increasing returns to scale, externalities,…), ideally one should be able to model congestion for convex and nonconvex technologies alike. Reinforcing the previous argument, it is obvious that ray disposability is of little use in nonconvex

[3] Since the proposed S-disposal assumption limits the extent of strong disposability but does not comply with any local notion in a mathematical sense, we opt for the adjective limited.

production models with indivisibilities.[4] Furthermore, the impact of nonconvexity on duality needs careful study. While already Jacobsen (1970) and Shephard (1974) point out that the cost function is convex (nonconvex) in the outputs under the assumption of a convex (nonconvex) technology, this result has been sharpened in Briec et al. (2004). These authors have shown that cost functions estimated on convex or nonconvex technologies only coincide in the constant returns to scale and single output case. Unfortunately, the issue of convexity in dual relations is widely ignored (see Kuosmanen 2003 for an exception). This calls for the development of nonconvex production technologies capable to model congestion, which is possible with the S-disposal assumption.

Third, ray disposal models of joint production have gained some popularity to explicitly model the trade-offs between good and bad outputs for the environment and to obtain shadow prices for these bads (see, e.g., Coggins and Swinton 1996).[5] Since these shadow prices are a direct consequence of the underlying duality relations, the enlargement of possible congestion notions may also have nonnegligible consequences in environmental modeling. If supported by the data, stronger notions of congestion may lead to higher shadow prices for bads than the ones hitherto obtained.[6] This issue is important given some pressing environmental challenges (e.g., global warming, fish stock decline, etc.).

This paper then proposes to model congestion using a new S-disposal assumption that allows defining multi-output technologies enveloping the data tighter than hitherto possible. The main reason for this methodological innovation is to reveal any congestion in production processes compatible with a minimal set of assumptions, in particular with or without convexity. This permits to model the full range of congestion notions defined in Färe and Svensson (1980) for the multi-output case from first principles.

The basic tool employed to characterize multi-output technologies and to detect all forms of congestion is the directional distance function. Being dual to the profit function (Luenberger 1995), it offers a general framework for economic analysis. This function has proven a useful tool in micro-economic theory as well as in applied production analysis (for example, it allows Chavas and Kim 2007 to shed new light on economies of scope from a primal viewpoint). However, in this contribution we need the directional distance function first and foremost because of the flexibility of its directional vector allowing to "look" for congestion in a precise and directed way. Given the theory-dependency of observations, the flexibility of this theoretical framework allows detecting forms of congestion that could hitherto not be observed

[4] A criticism on convexity in production theory (and economics in general) based on the importance of indivisibilities is developed in Scarf (1986). See also Hackman (2008).

[5] Murty et al. (2012) argue against this rather widespread use of ray disposability to model the relation between good and bad outputs. These authors explicitly combine a standard technology with good inputs and outputs with a residual generating technology which does not satisfy standard free disposal axioms. We ignore this particular application area focusing on the relation between good and bad outputs and focus on modeling congestion between good inputs and outputs instead.

[6] Since the acceptance/rejection of convexity may constitute a dividing line between economics and ecology (e.g., Dasgupta and Mähler 2003), it remains to be seen how trade-offs between good and bad outputs can be modeled without convexity.

using axiomatically founded technologies. Already Zhengfei and Oude Lansink (2003) convincingly illustrate how input congestion due to ray disposal of inputs is easier detected using some sub-vector distance function instead of a traditional one.

This paper unfolds as follows. Section 2 contains preliminary material on technologies, their subsets and underlying axioms. It also presents the new disposability axioms on technologies. Looking from a dual viewpoint, we focus on the fact that negative relative prices are linked to the congestion notion. Section 3 develops the notion of input directional distance functions on congested technologies and establishes the main duality result between the input directional distance function and a cost function allowing for negative prices. Furthermore, we show how to detect a lack of S-disposal and contrast this to the more traditional ray-disposable technologies and the different notions of congestion these technologies can reveal. We also outline a measure of congestion based upon the directional distance function. Thereafter, we indicate how observed finite data sets can or cannot be rationalized by a minimal technology compatible with S-disposal. The latter nonparametric test for the S-disposal hypothesis focuses on the cost function solely, thereby generalizing a well-known nonparametric test result in Varian (1984) (i.e., the Weak Axiom of Cost Minimization, WACM). It is motivated by the conviction that empirical production analysis must build upon minimal axioms.[7] This is in line with the recent upsurge in nonparametric models of consumption (e.g., Blundell 2005), characteristics models (for instance, Blow et al. 2008), etc. A final section concludes, discusses limitations, and offers directions for future research. In an effort not to overburden this already lengthy paper, an empirical illustration is made available in Appendix 1.

2 Technology: assumptions and definitions

2.1 Technology based upon traditional assumptions

We start by defining the notation used in this article. Let $\mathbb{R}^m_+$ be the nonnegative Euclidean m-dimensional orthant; for $x, u \in \mathbb{R}^m_+$ we denote $x \leq u \iff x_i \leq u_i$ $\forall i \in [m]$, where $[m]$ denotes the set $\{1, \ldots, m\}$.

A production technology transforming inputs $x = (x_1, \ldots, x_m)$ into outputs $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ can be characterized by the input correspondence $L : \mathbb{R}^n_+ \longrightarrow 2^{\mathbb{R}^m_+}$ where $L(y)$ is the set of all input vectors that yield at least y:

$$L(y) = \{x : x \text{ can produce } y\}, \tag{2.1}$$

and $2^{\mathbb{R}^m_+}$ the set of all subsets of $\mathbb{R}^m_+$.

Throughout this paper, we assume the input correspondence satisfies the following regularity properties (see Hackman 2008; Jacobsen 1970; McFadden 1978):

$L1$: $\forall y \geq 0$ with $y \neq 0$: $0 \notin L(y)$ and $L(0) = \mathbb{R}^m_+$.

7 Fuss et al. (1978, p. 223) state: "Given the qualitative, nonparametric nature of the fundamental axioms, this suggests [ ] that the more relevant tests will be nonparametric, rather than based on parametric functional forms, even very general ones."

$L2$: $\forall x \in \mathbb{R}^m_+$: $\bigcap_{y\in\mathbb{R}^n_+} L(y) \cap (x - \mathbb{R}^m_+) = \varnothing$.
$L3$: $L(y)$ is closed $\forall y \in \mathbb{R}^n_+$.
In addition to the axioms of no free lunch and the possibility of inaction ($L1$), as well as the boundedness ($L2$) and closedness ($L3$) of the inputs set, there are three other assumptions that we sometimes invoke on the input correspondence:
$L4$: $L(y)$ is a convex set $\forall y \in \mathbb{R}^n_+$.
$L5$: If $x \in L(y)$, then $\lambda x \in L(y)$, $\forall \lambda \geq 1$.
$L6$: Let $u \in \mathbb{R}^m_+$. If there exists a $x \in L(y)$ with $u \geq x$, then $u \in L(y)$.[8]

Assumption $L4$ postulates convexity of the input correspondence. This is useful to provide a dual interpretation through the cost function and in empirical applications of nonparametric technologies (e.g., Varian 1984). Assumption $L5$ postulates ray (or weak) disposability of the inputs, while axiom $L6$ imposes the more traditional assumption of strong (or free) disposal of inputs. A convex, ray disposable technology satisfying $L5$ but failing $L6$ is congested in the sense of Färe and Grosskopf (1983).[9] Note that $L4$ is not indispensable, since there exist nonparametric nonconvex technologies solely based upon the free disposal assumption $L6$ (e.g., Briec et al. 2004).[10]

To measure efficiency, it is convenient to distinguish between certain subsets of the input set $L(y)$. In particular, two subsets denoting production units on the boundary prove useful. For all $y \in \mathbb{R}^n_+$, the efficient subset is defined by:

$$E(y) = \{x \in L(y) : u \leq x \text{ and } u \neq x \Rightarrow u \notin L(y)\}. \tag{2.2}$$

The weak efficient subset is written as:

$$W(y) = \{x \in L(y) : u < x \Rightarrow u \notin L(y)\}. \tag{2.3}$$

2.2 The new *S*-disposal assumption

2.2.1 Congestion, S-disposability assumption and S-congestion

We start with a more precise definition of congestion. Transposing Färe and Svensson (1980) from the single to the multiple output case, Färe and Grosskopf (1983: 264) define monotone output-limitational (MOL) congestion as follows:

Definition 2.1 For all $y \in \mathbb{R}^n_+$, we say that the input set $L(y)$ is *MOL*-congested if for some $x \in L(y)$, $\exists u \geq x$ such that $u \notin L(y)$.

This means that a technology is MOL-congested if it fails the free disposal assumption. For instance, a ray disposable technology is MOL-congested (Färe and Grosskopf 1983).

[8] Using negation, one can easily see that this formulation of free disposability is logically equivalent with $\forall x \in L(y) : u \geq x \Rightarrow u \in L(y)$, the latter being somewhat more common.

[9] Kuosmanen (2005) shows that this traditional specification fails convexity, but that a revised specification is convex.

[10] See also First et al. (1993) and Hackman (2008) for alternative nonconvex technologies.

We define the following notation used throughout this contribution. Let $I \subset [m]$, then

$$x \leq_I u \iff \begin{cases} x_i \geq u_i & \text{if } i \in I \\ x_i \leq u_i & \text{else} \end{cases} \tag{2.4}$$

Moreover:

$$x <_I u \iff \begin{cases} x_i > u_i & \text{if } i \in I \\ x_i < u_i & \text{else} \end{cases} \tag{2.5}$$

Of course, if $-x \leq_I -u$ we denote $x \geq_I u$.

We can now define a new disposability assumption for the inputs. Denote by $2^{[m]}$ the set of all subsets of $[m]$. Remark that $\varnothing \in 2^{[m]}$ by definition.

Definition 2.2 Let $S \subset 2^{[m]}$. Let $u \in \mathbb{R}^m_+$ and $y \in \mathbb{R}^n_+$. The input correspondence $L(y)$ satisfies the S-disposal assumption if the following holds true: if for every $I \in S$ there exists a $x_I \in L(y)$ with $u \geq_I x_I$, then $u \in L(y)$.

Notice that if $S = \{\varnothing\}$, then we retrieve the standard vector inequality and the S-disposal assumption reduces to the standard free disposability assumption which can be seen easily by comparison with axiom $L6$. This S-disposal assumption is a kind of weakening of the usual strong or free disposal assumption. A technology which fails the strong disposal assumption may satisfy S-disposal assumption for a given S. Inversely, an S-disposable technology may violate the standard free disposability assumption depending on S.

Definition 2.3 Let $S \subset 2^{[m]}$. For all $y \in \mathbb{R}^n_+$, the input correspondence $L(y)$ satisfies a minimal S-disposability assumption if:

(a) $L(y)$ satisfies the S-disposal assumption, and
(b) $\nexists\, S' \subset S$ with $S' \neq S$ such that $L(y)$ satisfies the S'-disposal assumption.

Basically, the free disposal assumption is limited by combining it with a particular partial reversion of free disposal. Another way of interpreting this new definition is that we reformulate the traditional strong input disposability assumption as a "local" (in the sense of limited) rather than a global property (following the concerns expressed by Lau 1974).

Under weak disposability assumptions, any given input vector can be expanded along a ray through the origin. Consequently, there is no upper bound to wasting inputs, which seems a rather implausible assumption. By contrast, the more input dimensions are subjected to these particular, partial reversions of free disposability defined by the S-disposal assumption, the more the traditional free disposability assumption gets limited and thus weakened. Indeed, Definitions 2.2 and 2.3 imply that the larger the collection S is the more difficult one can dispose off inputs. In general, these definitions can account for cases where there is a simultaneous lack of free disposability in all dimensions, but it is also possible to define this lack independently in several dimensions. In conclusion, the S-disposal assumption allows accounting for upper bounds to the wasting of inputs that may well exist in certain data configurations.

Let us introduce the following convex cone:

$$K_I = \left\{x \in \mathbb{R}^m : x \geq_I 0\right\}. \tag{2.6}$$

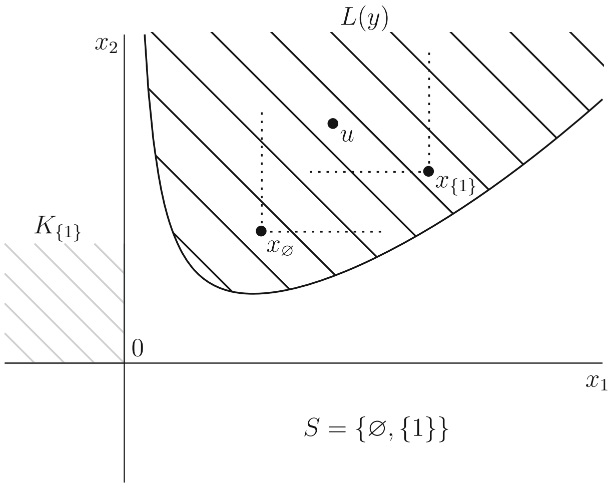

Fig. 1 The case $S = \{\varnothing, \{1\}\}$ on an input set

By definition, the nonnegative Euclidean orthant can be expressed as follows:

$$K_\varnothing = \mathbb{R}^m_+. \tag{2.7}$$

Definitions 2.2 and 2.3 can be illustrated in Figs. 1, 2 and 3. In Fig. 1, the input correspondence satisfies the minimal S-disposal assumption with $S = \{\varnothing, \{1\}\}$. For an arbitrary u, if there is some $x_\varnothing$ that classically dominates u and some $x_{\{1\}}$ that "{1}-dominates" u, then $u \in L(y)$. For a given configuration of observations, this serves to construct an input set where wasting the first input implies an additional opportunity cost in terms of the second input dimension. However, the reverse dependency between input dimensions does not hold.

In Fig. 2, there is lack of disposability in x_1 and x_2, but not in both dimensions simultaneously. Thus in this case, the input correspondence satisfies minimal S-disposability with $S = \{\varnothing, \{1\}, \{2\}\}$. By contrast, in Fig. 3 we show a potential example of the case $S = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$, which displays a reverse dependency between all input dimensions. This leads to an input set where the law of diminishing returns prevails in any input.[11]

Example 2.1 We consider the example proposed by (Färe and Jansson 1976, p. 410). Suppose that $m = 2$ and $n = 1$ and let the technology be defined by:

$$T = \left\{(x_1, x_2, y) \in \mathbb{R}^3_+ : y \leq \phi(x_1, x_2)\right\},$$

[11] The figure also reminds us about the possibility that there may be one or several bliss points where production is maximal. However, to clearly discern such a case one would need an approach also considering the output dimensions rather than just focusing on the input dimensions alone.

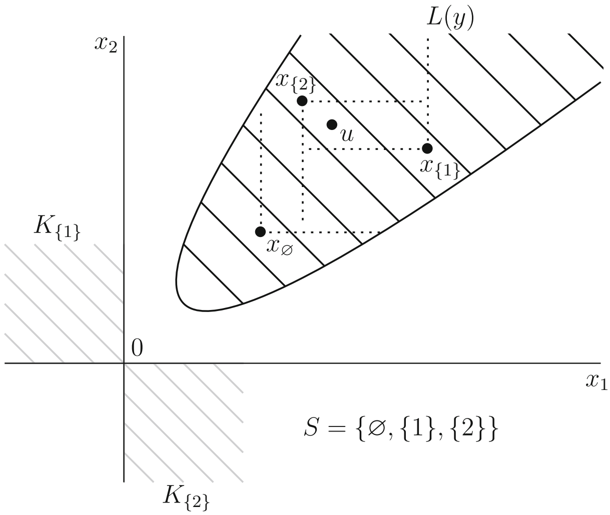

Fig. 2 The case $S = \{\varnothing, \{1\}, \{2\}\}$ on an input set

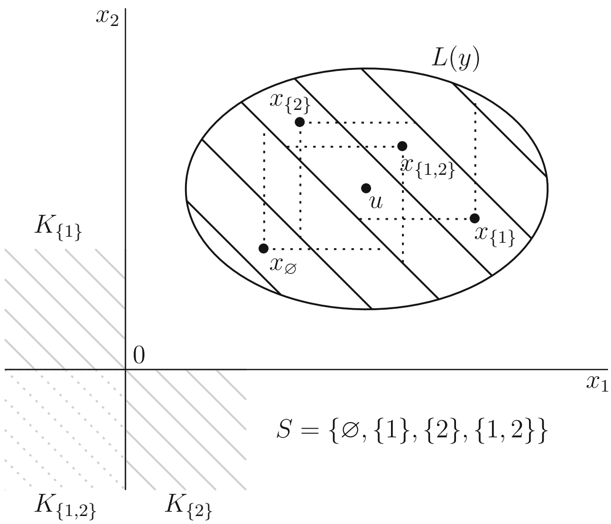

Fig. 3 The case $S = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$ on an input set

where

$$\phi(x_1, x_2) = \begin{cases} \alpha[(1-\delta)(x_1 - \beta_2 x_2)^{-q} \\ \quad +\delta(x_2 - \beta_1 x_1)^{-q}]^{-1/q} & \text{if } \min\{x_1 - \beta_2 x_2, x_2 - \beta_1 x_1\} \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

with the parameters restricted as $\alpha > 0$, $\delta \in]0, 1[$, $q \in [-1, +\infty[$, $\beta_1, \beta_2 \in [0, +\infty[$, $\beta_1\beta_2 < 1$. Suppose that $y > 0$. Then, by definition:

$$L(y) = \{(x_1, x_2) \in \mathbb{R}_+^2 : \phi(x_1, x_2) \geq y\}.$$

Consider the following cases:

(i) If $\alpha = 1$, $\delta = 0.5$, $\beta_1 = \beta_2 = 0.2$, $q = 0.2$, then for all $y > 0$ the input set satisfies the S-disposal assumption with $S = \{\emptyset, \{1\}, \{2\}\}$ (see Färe and Jansson 1976, Figure 1 on p. 410). However, it fails the S-disposal assumption if either $\{1\}$ or $\{2\}$ does not belong to S.

(ii) If $\alpha = 1$, $\delta = 0.5$, $\beta_1 = 0.2$ but $\beta_2 = 0$, $q = 0.2$, then for all $y > 0$ the input set satisfies the S-disposal assumption with $S = \{\emptyset, \{1\}\}$ (see Färe and Jansson 1976, Figure 2 on p. 411). However, it fails the free disposal assumption. This input set is freely disposable in the second input dimension, but it is congested in the first input dimension. Hence, the technology satisfies the S-disposability assumption with $S = \{\emptyset, \{1\}\}$.

Example 2.2 Suppose that $m = 2$ and $n = 1$ and let the technology be defined by:

$$T = \{(x_1, x_2, y) \in \mathbb{R}_+^3 : y \leq \alpha x_1 x_2 ({x_2}^2 + \beta^2)^{-1}\},$$

where α and β are two positive parameters. Suppose that $y > 0$. Then, by definition:

$$\begin{aligned} L(y) &= \{(x_1, x_2) \in \mathbb{R}_+^2 : \alpha x_1 x_2 ({x_2}^2 + \beta^2)^{-1} \geq y\} \\ &= \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 \geq y\alpha^{-1} {x_2}^{-1} ({x_2}^2 + \beta^2)\}. \end{aligned}$$

This input set is freely disposable in the first input dimension, but it is congested in the second input dimension. Hence, the technology satisfies the S-disposability assumption with $S = \{\emptyset, \{2\}\}$.

The previous examples might suggest that S-disposability can always be imposed by carefully selecting the set S. Unfortunately, this is not the case as can be seen, e.g., when considering an input set that is not path connected. However, if the input set is convex, then S-disposability with $S = 2^{[m]}$ is always satisfied as suggested by Figs. 1, 2 and 3.

To study this new disposal assumption from a dual standpoint, we introduce the cost function $C : \mathbb{R}^m \times \mathbb{R}_+^n \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by:

$$C(p, y) = \begin{cases} \inf_x \{p.x : x \in L(y)\} & \text{if } L(y) \neq \emptyset, \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that this definition allows to take into account negative prices which are specifically linked to congested technologies.

The following proposition studies the properties of the S-disposal assumption.

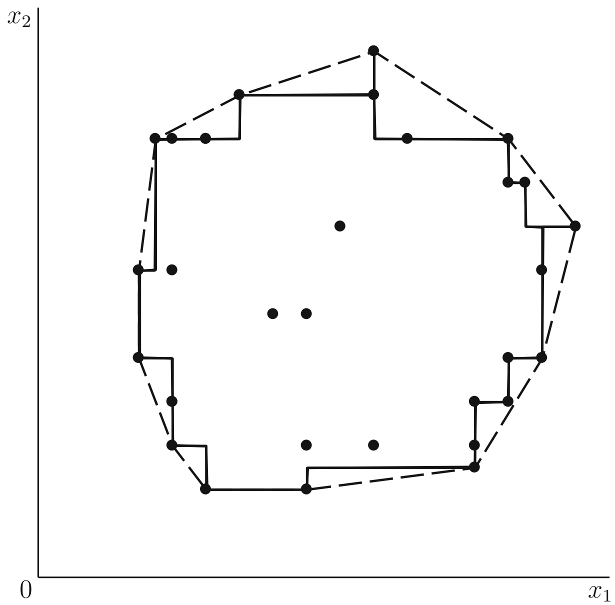

Fig. 4 Convex and nonconvex input correspondence

Proposition 2.1 *Let L be an input correspondence satisfying $L1$–$L3$. For all $y \in \mathbb{R}^n_+$, if $L(y)$ is nonempty then we have the following properties:*

(a) *Let S and S' be two collections of subsets of $[m]$ such that $S \subset S'$. If $L(y)$ satisfies the S-disposal assumption, then it also satisfies the S'-disposal assumption.*
(b) *$L(y)$ satisfies the S-disposal assumption if and only if:*

$$L(y) = \bigcap_{I \in S} (L(y) + K_I) .$$

Note that the proofs of all propositions are in Appendix 2. Part (a) states that if an input set satisfies S-disposal of a certain dimensionality, then the same technology is compatible with S'-disposal for every set S' containing the initial collection S. Part (b) characterizes an S-disposal input correspondence in terms of an intersection of subsets constructed by means of the cones (2.6).

Although this result may seem rather trivial, it actually turns out to be very useful as an alternative for obtaining the input correspondence $L(y)$. Indeed, since no assumptions are made concerning convexity, this result also holds true under nonconvexity. To give just one example of its application, result (b) can be used for defining and reconstructing a nonconvex hull technology in an indirect way. Figure 4 shows the nonconvex hull (solid line) and the convex hull (dashed line) input correspondence of a set of observations with two inputs.

The following proposition extends the results of Proposition 2.1 to a convex input correspondence. In particular, we provide a dual characterization of the S-disposability notion. But, before doing so, we first define the notion of J-congested price.

Definition 2.4 Suppose that the input set satisfies the S-disposal assumption. For all $J \in S$, we say that an input price p is J-congested if

$$p \in K_J \cap \left(\mathbb{R}^m \backslash \bigcup_{I \in S \backslash \{J\}} K_I \right).$$

Example 2.3 To illustrate this definition, suppose that $m = 2$ and $S = \{\varnothing, \{1\}\}$. A price vector is then $\{1\}$-congested if

$$p \in K_{\{1\}} \cap \left(\mathbb{R}^2 \backslash K_\varnothing\right) = (\mathbb{R}_- \times \mathbb{R}_+) \cap \left(\mathbb{R}^2 \backslash \mathbb{R}^2_+\right).$$

Equivalently, $p_1 < 0$ and $p_2 \geq 0$.

In general, if $S = \{\varnothing, J\}$, a price is J-congested if $p_j < 0$ for all $j \in J$ and $p_i \geq 0$ for all $i \notin J$.

Proposition 2.2 *Let L be an input correspondence satisfying $L1$–$L3$. Moreover, assume that $L4$ holds. For all $y \in \mathbb{R}^n_+$, if $L(y)$ is nonempty then we have the following properties:*

(a) *$L(y)$ satisfies the S-disposal assumption if and only if*

$$L(y) = \left\{ x \in \mathbb{R}^m : p.x \geq C(p, y), p \in \bigcup_{I \in S} K_I \right\}.$$

(b) *There exists a collection S that contains $\varnothing$ such that $L(y)$ satisfies the minimal S-disposal assumption.*
(c) *Assume that $L(y)$ satisfies the S-disposal assumption. The S-disposability of $L(y)$ is minimal, if and only if for all $J \in S$ there exists some J-congested price p such that $C(p, y) > -\infty$.*

Intuitively stated, a convex input set satisfying S-disposal can be enveloped by a cost function for proper prices. More precisely, if one defines a minimal S-disposal input set, then a support function can be defined with negative prices corresponding to the subset of all congesting input dimensions. This result constitutes the basis for the duality result developed in Sect. 3 below.

We are now ready to define a new, more general congestion notion:

Definition 2.5 Let L be an input correspondence and let S be a collection of subsets in $[m]$ that contains $\varnothing$. Let $y \in \mathbb{R}^n_+$. $L(y)$ is said to be S-congested if it is nonempty and fails the S-disposal assumption.

Definition 2.5 provides a strict definition of S-congestion by assuming that there does not exist a stronger S-disposal assumption holding over the input correspondence. In particular, this means that a S-congested technology is such that:

$$L(y) \neq L_\varnothing(y) = L(y) + \mathbb{R}^m_+. \tag{2.8}$$

This can be viewed as a transposition of the earlier definition of MOL-congestion in terms of S-disposability. This facilitates comparisons among concepts.

The next result establishes a characterization of S-congested technologies.

Proposition 2.3 *Let L be an input correspondence that satisfies $L1$-$L3$. Let S be a collection of subsets in $[m]$ that contains $\varnothing$. For all $y \in \mathbb{R}^n_+$, if $L(y)$ is nonempty then we have the following properties:*

(a) *Assume that $L(y)$ is S-congested. For all $S' \subset S$ with $S' \neq \varnothing$, $L(y)$ is S'-congested.*
(b) *Assume that $L(y)$ satisfies the minimal S-disposability assumption, then for all $S' \subset S$ with $S' \neq S$, $L(y)$ is S'-congested.*
(c) *Assume that $L4$ holds. $L(y)$ is S-congested if and only if there exists $J \notin S$, and some J-congested price vector p_J such that $C(p_J, y) > -\infty$.*

Parts (a) and (b) of Proposition 2.3 state that if an input set is S-congested in terms of a certain dimensionality, then the same technology is S'-congested for every proper subset S' of the initial collection S.

2.2.2 Boundaries and bounds on S-congested technologies

It remains an open question how to detect congestion from the structure of the input correspondence. To answer this question, it is useful to introduce the concept of a congestion frontier. Therefore, the following definition identifies a subset that is not efficient, but that is a part of the boundary of a congested input correspondence.

Definition 2.6 Let L be an input correspondence and let $I \subset [m]$. For all $y \in \mathbb{R}^n_+$, we call the I-congested boundary the subset:

$$E_I(y) = \{x \in L(y) : u \leq_I x \text{ and } u \neq x \Rightarrow u \notin L(y)\}.$$

We call the I-weakly congested boundary the subset:

$$W_I(y) = \{x \in L(y) : u <_I x \Rightarrow u \notin L(y)\}.$$

Let S be a collection of subsets of $[m]$. We call the S-congested boundary and weakly S-congested boundary, respectively,

$$E_S(y) = \bigcup_{I \in S} E_I(y) \quad \text{and} \quad W_S(y) = \bigcup_{I \in S} W_I(y).$$

Example 2.4 In Example 2.2, consider the curve defined by equation $x_1 = y\alpha^{-1}x_2^{-1}(x_2{}^2 + \beta^2)$. The minimum point of the curve is achieved at $x_2 = \beta$. With $S = \{\varnothing, \{2\}\}$, we have $E_{\{2\}}(y) = W_{\{2\}}(y) = \{(y\alpha^{-1}x_2^{-1}(x_2^2 + \beta^2), x_2) : x_2 \geq \beta\}$ and $E_\varnothing(y) = W_\varnothing(y) = \{(y\alpha^{-1}x_2^{-1}(x_2^2 + \beta^2), x_2) : 0 \leq x_2 \leq \beta\}$.

Proposition 2.4 *Let L be an input correspondence satisfying $L1$–$L3$ and S be a collection of subsets of $[m]$ that contains $\varnothing$.*

(a) *The subset $W_S(y)$ is closed.*[12]
(b) *The input set $L(y)$ is S-congested if and only if there exists some $J \notin S$ such that the subset $E_J(y)$ is nonempty.*
(c) *The input set $L(y)$ satisfies a minimal S-disposal assumption if and only if for every $J \in S$ the subset $E_J(y)$ is nonempty.*
(d) *Assume that L4 holds. The input set $L(y)$ is S-congested if and only if there exists some $J \notin S$, $x_J \in W_J(y)$ and some J-congested price vector p_J such that $C(p_J, y) > -\infty$ and $p_J.x_J = C(p_J, y)$.*

In Propositions 2.2, 2.3 and 2.4 we have developed some connections between the S-congestion concept and the cost function. Obviously, when the free disposability assumption holds, then $C(p, y) > -\infty \iff p \geq 0$. However, the S-disposal assumption condition $p_J \in K_J \cap \left(\mathbb{R}^m \setminus \bigcup_{I \in S\setminus\{J\}} K_I\right)$ does not warrant that $C(p_J, y) > -\infty$. In fact, to obtain a similar property on the cost function, we introduce the S-bounded concepts. When the usual free disposability assumption holds, since $L(y) \subset \mathbb{R}^m_+$, then the input correspondence is $\varnothing$-bounded.

Definition 2.7 Let L be an input correspondence and S be a collection of subsets of $[m]$ that contains $\varnothing$. For all $y \in \mathbb{R}^n_+$, the input set $L(y)$ is S-bounded if for all $I \in S$ there exists some $\bar{x}_I \leq_I x, \forall x \in \mathbb{R}^m_+$.

Obviously, an input set that satisfies the usual free disposal assumption is $\varnothing$-bounded, with $\bar{x}_\varnothing = 0$. We show in Proposition 2.5 below that the above Definition 2.6 is of particular interest in the context of defining empirical specifications (e.g., nonparametric) of technologies.

Proposition 2.5 *Let L an input correspondence that satisfies $L1$-$L3$. Let S a collection of subsets in $[m]$ that contains $\varnothing$. For all $y \in \mathbb{R}^n_+$, if $L(y)$ is nonempty then we have the following properties:*

(a) *If $L(y)$ is S-bounded and $\bigcup_{I \in S} I = [m]$, then $L(y)$ is compact.*
(b) *If $L(y)$ is S-bounded, then for every $S' \subset S$, with $S' \neq S$, it is S'-congested.*
(c) *Assume that L4 holds, if $L(y)$ is S-bounded, then there exists some $J \in S$ and a J-congested price p_J such that $C(p_J, y) > -\infty$.*

Part (b) of Proposition 2.5 states that if an input set is S-bounded in terms of a certain dimensionality, then the same technology is S'-congested for every proper subset S' of the initial collection S. This explains why congestion is so easily ignored. Assuming congestion is present in some dimensions of the true technology, then if one does not specify a general enough model one may miss detecting congestion in some of these dimensions. Thus, it is key to start by specifying a general model capturing any congestion present in all input dimensions. If no congestion is found in all input dimensions, then one can look for subsets of inputs suffering from congestion. But, if one starts with a specific model for a subset of inputs, then one may miss detecting congestion in other or all inputs. In the limit, if one is unwilling to impose a model

[12] As pointed out by an anonymous referee, in general the efficient subset is not closed (see for instance Arrow et al. 1953).

capable to capture congestion, then no congestion can appear at all. By analogy, one could also say that if one looks for OP-congestion one may end up finding MOL-congestion. But, when looking for MOL-congestion only, one is never able to find any OP-congestion.

The following example shows how to compute a congested cost function.

Example 2.5 Consider the minimization problem

$$\min_x \left\{ p_1 x_1 + p_2 x_2 : \phi(x_1, x_y) \geq y \geq 0 \right\},$$

defined from the technology introduced in Example 2.1 which is equivalent to

$$\min_x \Big\{ p_1 x_1 + p_2 x_2 : \alpha[(1-\delta)(x_1 - \beta_2 x_2)^{-q} + \delta(x_2 - \beta_1 x_1)^{-q}]^{-1/q} \geq y, \\ \min\{x_1 - \beta_2 x_2, x_2 - \beta_1 x_1\} \geq 0 \Big\}.$$

Denote $u = \begin{pmatrix} 1 & -\beta_2 \\ -\beta_1 & 1 \end{pmatrix} x$. Then $x = \frac{1}{1-\beta_1\beta_2} \begin{pmatrix} 1 & \beta_2 \\ \beta_1 & 1 \end{pmatrix} u$. Consequently, the minimization problem can be rewritten as

$$\min_u \Big\{ \frac{1}{1-\beta_1\beta_2} [(p_1 + \beta_1 p_2) u_1 + (p_2 + \beta_2 p_1) u_2] : \\ \alpha[(1-\delta) u_1^{-q} + \delta u_2^{-q}]^{-1/q} \geq y, u_1, u_2 \geq 0 \Big\}.$$

If $p_1 + \beta_1 p_2 > 0$ and $p_2 + \beta_2 p_1 > 0$, we obtain from the first order Karush–Kuhn–Tucker conditions the optimal solution

$$\bar{u}_1 = \frac{y(1-\delta)^s (p_2 + \beta_2 p_1)^s}{\alpha[\delta^s (p_1 + \beta_1 p_2)^{s-1} + (1-\delta)^s (p_2 + \beta_2 p_1)^{s-1}]^{-\frac{1}{q}}}$$

and

$$\bar{u}_2 = \frac{y\delta^s (p_1 + \beta_1 p_2)^s}{\alpha[\delta^s (p_1 + \beta_1 p_2)^{s-1} + (1-\delta)^s (p_2 + \beta_2 p_1)^{s-1}]^{-\frac{1}{q}}},$$

with $s = 1/(1+q)$. Using the relation between x and u yields the solution $\bar{x}$ of the cost minimization problem. Notice that the conditions $p_1 + \beta_1 p_2 > 0$ and $p_2 + \beta_2 p_1 > 0$ allow computing the cost function for possible nonpositive price vectors.

Example 2.6 Consider Examples 2.2 and 2.4 and assume that $p_1 > 0$ and $p_2 < 0$. From the results above

$$C(p, y) = \inf_x \{p.x : x \in W_{\{2\}}(y)\}.$$

Since $p_2 < 0$, the constraint $x_2 \geq \beta$ is not binding. Thus

$$C(p, y) = \inf_x \{p_1 y \alpha^{-1} x_2^{-1} \left(x_2{}^2 + \beta^2\right) + p_2 x_2\}.$$

We obtain the first order condition:

$$2p_1 y \alpha^{-1} x_2{}^2 + 2p_2 x_2{}^2 - p_1 y \alpha^{-1} \left(x_2{}^2 + \beta^2\right) - p_2 x_2{}^2 = 0$$

leading to the optimal solution

$$\bar{x}_2 = \sqrt{\frac{p_1 y \alpha^{-1} \beta^2}{p_1 y \alpha^{-1} + p_2}} \quad \text{and} \quad \bar{x}_1 = p_1 y \alpha^{-1} \bar{x}_2^{-1} (\bar{x}_2^2 + \beta^2),$$

which yields the cost function.

3 Duality between technology and cost function based on *S*-disposability: a new result

Luenberger (1992) introduced the so-called benefit function in consumer theory. Chambers et al. (1996) have transposed this measure in the context of production theory by defining the input directional distance function. This input directional distance function characterizes technology and provides a useful tool in efficiency and productivity measurement because it generalizes the traditional input distance function and thus also the radial efficiency measure. This input directional distance function is a special case of the directional distance function that itself is dual to the profit function (see Luenberger 1995). Therefore, the use of directional distance functions offers the most general framework.

3.1 Directional distance function and cost function on *S*-congested technologies: a duality result

The input directional distance function $D_L : \mathbb{R}_+^{m+n} \times \mathbb{R}_+^m \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by:

$$D_L(x, y; g) = \sup\{\delta : x - \delta g \in L(y)\}. \tag{3.1}$$

Note that $g \in K_\varnothing = \mathbb{R}_+^m$ in the definition above which holds for a technology that satisfies the strong disposability assumption.

Following the traditional duality result in Jacobsen (1970) or McFadden (1978) between cost function and input distance function, Luenberger (1992) and Chambers et al. (1996) have more recently developed formulations in terms of cost function and input directional distance function. Thus, one can state a duality result making a link between the input directional distance function and the cost function on an input set $L(y)$ satisfying the strong disposability assumption.

Proposition 3.1 *Let L be an input correspondence satisfying $L1$-$L3$ and $L6$. Assume that $g \neq 0$ and let $(x, y) \in \mathbb{R}_+^{m+n}$ such that $D_L(x, y; g) > -\infty$. We have the following properties:*

(a) *If $L4$ (convexity) holds:*

$$D_L(x, y; g) = \inf_p \{p.x - C(p, y) : p.g = 1, p \geq 0\}. \tag{3.2}$$

(b) *Let p be a nonnegative input price vector. Assuming that $L4$ holds, we have:*

$$C(p, y) = \inf_x \{p.x - p.gD_L(x, y; g)\}. \tag{3.3}$$

Apart from this traditional duality relationship, a weaker duality result between the cost function and the ray (or weakly) disposable input distance function is available in the literature (e.g., Shephard 1974) whereby some (but not all) prices are allowed to be negative:[13]

Proposition 3.2 *Let L be an input correspondence satisfying $L1$-$L3$ and $L5$. Assume that $g \neq 0$ and let $(x, y) \in \mathbb{R}_+^{m+n}$ such that $D_L(x, y; g) > -\infty$. We have the following properties:*

(a) *If $L4$ (convexity) holds, then:*

$$D_L(x, y; g) = \inf_p \{p.x - C(p, y) : p.g = 1\}. \tag{3.4}$$

(b) *Let p be an input price vector having some negative components. Assuming that $L4$ holds, we have:*

$$C(p, y) = \inf_x \{p.x - p.gD_L(x, y; g)\}. \tag{3.5}$$

This is an immediate consequence from the fact that a convex set is the intersection of its supporting hyperplanes.

Now, we extend the directional distance function and its properties to account for input sets satisfying the S-disposal assumption combined with a possible nonpositive direction vector g.

Proposition 3.3 *Let L be an input correspondence satisfying $L1$-$L3$. Assume moreover that $L(y)$ satisfies the S-disposal assumption. Assume that $g \neq 0$ and let $(x, y) \in \mathbb{R}_+^{m+m}$ such that $D_L(x, y; g) > -\infty$. We have the following properties:*

(a) *If $L4$ (convexity) holds, then:*

$$D_L(x, y; g) = \inf_p \left\{ p.x - C(p, y) : p.g = 1, p \in \bigcup_{I \in S} K_I \right\}. \tag{3.6}$$

[13] Also McFadden (1978, 60) anticipates the use of negative prices and maintains that duality results can be preserved under these circumstances.

(b) *Let $p \in K_I$ with $I \in S$ be an input price vector having some negative components. Assume that L4 holds, we have:*

$$C(p, y) = \inf_x \{p.x - p.gD_L(x, y; g) : x \in L(y)\}. \tag{3.7}$$

Property (a) extends the results by Luenberger (1992) and Chambers et al. (1996) in the context of an input correspondence that may fail both the strong and the weak disposability assumptions. The converse results expressing the cost function with respect to the directional distance function is stated in (b). This duality result considerably weakens current duality results imposing strong disposability (Proposition 3.1) and weak disposability (Proposition 3.2), which allow some (but not all) prices to be negative. Otherwise stated, this proposition shows that S-disposal of inputs is a necessary and sufficient condition for the input directional distance function to characterize technology. This substantially weakens the existing result on the importance of ray disposal in the inputs for the traditional input distance function to characterize technology.

This new duality result is illustrated in Fig. 5 for the case $S = \{\varnothing, \{1\}\}$. Since the first input is clearly congested, it receives a negative price and the cost function ends up having a positive rather than a negative slope.

Notice that when modeling the trade-offs between good and bad outputs for the environment, the S-disposal assumption may well lead to more negative prices for the bads than the traditional ray disposal assumption currently employed in this literature. This may imply that current estimates for the economic harm of bads are systematically underestimated.

In principle it is possible to relax the convexity assumption. Under nonconvexity, the duality result in Proposition 3.3 would only hold locally (similar to the local duality result in, e.g., Briec et al. 2004). However, under nonconvexity Proposition 3.2 would fail to hold, since ray disposal seems to be of little use without convexity (see Introduction). Note again that while the cost function is nondecreasing in the outputs, cost functions estimated on convex (nonconvex) technologies are furthermore convex (nonconvex) in the outputs (see Jacobsen 1970; Shephard 1974).

3.2 *S*-congested and ray-disposability congested technologies: a comparison

It should be clear by now that when the input set satisfies free disposal, then it also satisfies S-disposal assumptions. But, the converse is not necessarily true. The same applies to weak disposal assumptions: an input set satisfying weak disposability assumptions also satisfies S-disposal assumptions, but the converse need not be true.

An input set that is weakly disposable can be employed to detect MOL-congestion whereby increasing some inputs decreases outputs (or decreasing inputs increases outputs). An input set satisfying S-disposal assumptions can also detect hypercongestion. This subsection clarifies the link between both approaches to congestion modeling.

Proposition 2.2 has a direct implication for the ray disposability notion. When technology is convex and MOL-congested (see Färe and Grosskopf 1983), then there exists a collection S such that it satisfies a minimal S-disposal assumption and one obtains negative marginal rates of substitution corresponding to the lack of free disposability.

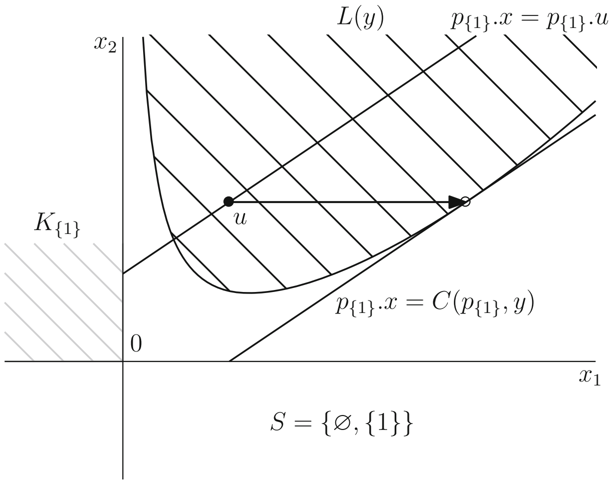

Fig. 5 Directional distance function and duality with $I = \{1\}$ and $S = \{\varnothing, \{1\}\}$

Proposition 3.4 *Let L be an input correspondence satisfying L1–L4. For all $y \in \mathbb{R}^n_+$, if $L(y)$ satisfies L5, but not L6, then we have the following properties:*

(a) *There exists S that contains ∅ such that $L(y)$ satisfies a minimal S-disposal assumption.*
(b) *There exists S such that for every $J \in S$, there exists some input J-congested price vector p such that $C(p, y) > -\infty$.*

Proposition 3.4 characterizes ray disposability in the inputs as a special case of minimal S-disposability. Part (a) states that any weakly disposable technology can be re-interpreted as an S-disposable technology, but not the converse. Part (b) claims that a weakly disposable technology can always be characterized via the support function of its input set. An input set is then ray disposable if there exists a price vector containing some negative prices such that the cost function is bounded.

This results is illustrated in Fig. 6. A weakly disposable input set with backward bending rays can also be reconstructed as an input set with S-disposal in both inputs (just as the case $S = \{\varnothing, \{1\}, \{2\}\}$ depicted in Fig. 2).

The following corollary establishes a link between S-congestion and the notion of MOL-congestion as defined by Färe and Grosskopf (1983).

Proposition 3.5 *Let L be an input correspondence satisfying L1-L4. For all $y \in \mathbb{R}^n_+$, if $L(y)$ satisfies L5, but not L6, then we have the following properties:*

(a) *There exists S that contains ∅ such that $L(y)$ is S-congested.*
(b) *There exists a collection S, $J \notin S$ and some J-congested price vector p such that $C(p, y) > -\infty$.*

Thus, for any input set satisfying a certain ray disposal hypothesis, one can always find a corresponding S-congestion assumption that is compatible with the data. Furthermore, the former input set can be reconstructed via a corresponding cost function, just as it is the case for S-congested technologies.

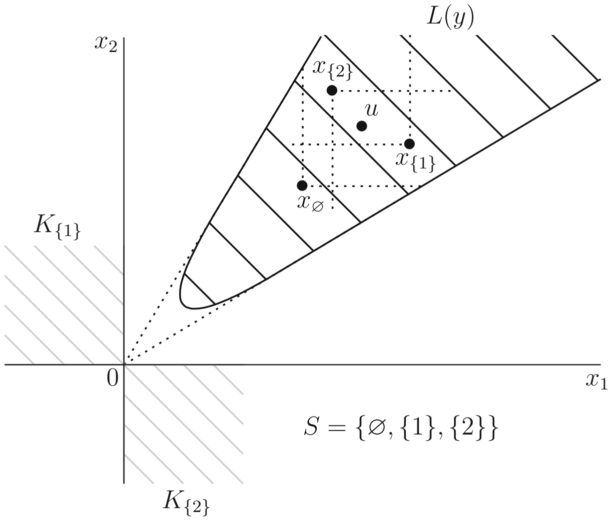

Fig. 6 Weakly disposable technologies and S-congestion

In the following we show that S-congestion can be viewed as a more flexible concept because it can model several forms of congestion as defined in Färe and Svensson (1980). Indeed, in addition to MOL-congestion it also allows to model output prohibitive (OP) congestion and all of its variations in general multi-output technologies. Let J be a finite subset of $[n]$. In the following we denote $\mathbb{R}^J_+ = \{y \in \mathbb{R}^n_+ : y_j = 0, \forall j \notin J\}$. By definition, we have $\mathbb{R}^J_+ = \mathbb{R}^n_+ \cap K_J$, where K_J is the convex cone of $\mathbb{R}^n$ constructed following Eq. (2.6).

Definition 3.1 For all $y \in \mathbb{R}^n_+$, an input set $L(y)$ is OP-congested for the index set J if for all $y \in \mathbb{R}^J_+ \backslash \{0\}$ the input set $L(y)$ is J-bounded.

This definition generalizes OP-congestion as defined for single-output technologies in Färe and Svensson (1980) to the multiple output case.

The next result establishes that a ray disposable technology cannot satisfy OP-congestion.

Proposition 3.6 *Let L be an input correspondence satisfying $L1$-$L4$. For all $y \in \mathbb{R}^n_+$, if $L(y)$ satisfies $L5$, then the technology does not satisfy OP-congestion.*

By contrast, a S-congested technology may exhibit OP-congestion. This is established in the next example.

Example 3.1 Suppose that $m = n = 1$ and that there exists a continuous function $\phi : \mathbb{R}_+ \Rightarrow \mathbb{R}_+$ such that $T = \{(x, y) \in \mathbb{R}^2_+ : y \leq \phi(x)\}$. Suppose moreover that: (a) there exists a unique maximum x^* of ϕ, (b) $\lim_{x \to +\infty} \phi(x) = 0$, and (c) $\phi(0) = 0$. From (a), (b) and (c), this technology satisfies $L1$-$L4$ and is clearly OP-congested. Suppose that $x, x' \in L(y)$ and that $u \geq x$ and $u \leq x'$. Since there is a unique maximum and ϕ is continuous, we have $\phi(u) \geq y$. Consequently, $u \in L(y)$ and we conclude that $L(y)$ satisfies a $\{\varnothing, \{1\}\}$-disposal assumption. Moreover, it fails the free disposal assumption ($\varnothing$-disposal). Thus, $L(y)$ is $\varnothing$-congested.

One could easily extend this example to the multiple output case.

3.3 Directional distance function and congestion measurement

We are now interested in making the link between special cases of the input directional distance function, to be introduced below and the congestion concept. To study this relationship from the dual viewpoint we introduce the adjusted price correspondence $p : \mathbb{R}^{m+n}_+ \times \mathbb{R}^m_+ \longrightarrow 2^{\mathbb{R}^m}$ due to Luenberger (1995) and defined by:

$$p(x, y; g) = \arg\min \left\{ p.x - C(p, y) : p.g = 1, p \in \bigcup_{I \in S} K_I \right\}. \tag{3.8}$$

Notice that if the minimum is not achieved, then $p(x, y; g) = \varnothing$. For simplicity, we introduce the following notation:

$$L_\varnothing(y) = L(y) + K_\varnothing = L(y) + \mathbb{R}^m_+, \tag{3.9}$$

$$L_I(y) = L(y) + K_I, \tag{3.10}$$

$$L_S(y) = \bigcap_{I \in S} L_I(y). \tag{3.11}$$

Proposition 3.7 *Let L be an input correspondence satisfying $L1$–$L3$. For all $y \in \mathbb{R}^n_+$, we have the following properties:*

(a) *If $L(y)$ is S-congested, then there is $J \notin S$, $g_J \in K_J$ and $x \in W_J(y)$ such that $D_{L_S}(x, y; g_J) = 0$.*

(b) *If $L(y)$ satisfies a minimal S-disposal assumption, then for all $J \in S$, there are $g_J \in K_J$ and $x \in W_J(y)$ such that $D_{L_S}(x, y; g_J) = 0$.*

In the next proposition, the impact of adding convexity to axioms $L1$–$L3$ is analyzed.

Proposition 3.8 *Let L be an input correspondence satisfying $L1$-$L4$. For all $y \in \mathbb{R}^n_+$, we have the following properties:*

(a) *$L(y)$ is S-congested if and only if there exists $J \notin S$ and there are $g_J \in K_J$ and $x \in L(y)$ such that $p(x, y; g_J) \subset K_J$.*
(b) *$L(y)$ satisfies a minimal S-disposal assumption if and only if for all $J \in S$ there are $g_J \in K_J$ and $x \in L(y)$ such that $p(x, y; g_J) \subset K_J$.*
(c) *$L(y)$ is S-congested if and only if there exists $J \notin S$ and some $x \in L(y)$ such that $D_L(x, y; g_J) < D_{L_S}(x, y; g_J)$.*
(d) *The S-disposal assumption is minimal if and only if for all $J \in S$ and $x \in L(y)$: $D_L(x, y; g_J) < D_{L_S}(x, y; g_J)$.*

Remark that the properties above hold for the general case of a direction vector g_J possibly having some negative components. However, from a practical standpoint the direction g_J should be chosen such that the corresponding directional distance function value actually measures congestion and guarantees feasible solutions. For this, we introduce the following appropriate direction vector and corresponding congestion measure.

Definition 3.2 Let L be an input correspondence. Assume that $I \in S$, where S is a collection of subsets of $[m]$. Suppose that $I \neq \emptyset$ and let $g_I \in \mathbb{R}^m$ be a vector such that $g_{I_i} \leq 0$ if $i \in I$ and $g_{I_i} = 0$ otherwise. For a production combination $(x, y) \in T$, we define the I-congestion measure in the direction g_I as $DC_I(x, y; g_I) = D_{L_I}(x, y; g_I)$.

Note that the congestion measure $DC_I(x, y; g_I)$ evaluates eventual congestion in a component wise way per subset of S. This can be illustrated with the help of Fig. 3 showing the case where the set $S = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. This implies that one can measure $\{1\}$-congestion in the horizontal direction, $\{2\}$-congestion in the vertical direction, and $\{1, 2\}$-congestion into a proportional direction towards the northeast upper bound of the input set at the back.

This discussion can be linked again to Proposition 2.5 to illustrate the theory-dependency of empirical measurements and the need for a sufficiently general model. Even if the empirical data configuration would allow to generate an input set as depicted in Fig. 3, it is straightforward to ignore the potential severity of the congestion phenomenon. For instance, if one tries measuring $\{1, 2\}$-congestion and the data corroborate this hypothesis, then one knows that also some form of $\{1\}$- and $\{2\}$-congestion is present. By contrast, if one would start out by measuring $\{1\}$-congestion ($\{2\}$-congestion) instead and finding it, then nothing is implied about finding $\{2\}$-congestion ($\{1\}$-congestion) or $\{1, 2\}$-congestion in the data. This explains why congestion may remain unnoticed: even if congestion is initially present in some dimensions of the true technology, then it is always possible to ignore measuring congestion in some of these dimensions until eventually no congestion seems to appear at all. This may contribute to explaining its neglect in applied production theory.

So far congestion is treated as a particular severe form of technical inefficiency. This fits into a traditional decomposition of static input-oriented inefficiency whereby congestion and technical inefficiency are analyzed first, then eventually one evaluates inefficiencies related to returns to scale, and finally one defines allocative efficiency as closing the gap between this engineering perspective (combining congestion, technical and scale inefficiencies) and some ideal economic reference point (e.g, resulting from cost minimization). Obviously, one could define the order of this static decomposition otherwise which, e.g., could lead one to combine S-disposal with more specific returns to scale assumptions.[14]

[14] See, e.g., Färe and Grosskopf (2000) and McDonald (1996) who argue in favor and against the above mentioned static decomposition respectively. Note that this whole issue is solely based on economic tradition, not on any empirical evidence indicating what decision-makers find relevant information.

3.4 Testing for consistency with cost minimization

Suppose we are given some data on input–output vectors (x^j, y^j) and input prices p^j for all $j \in J$. Here we ask whether or not there exists a family of input sets $L(y)$ that can make sense of this observed behavior. It is possible to show that the existence of negative prices involves congestion in the general sense defined in this contribution. Following Varian (1984) we say that a family of input sets $L(y)$ c-rationalizes the data if x^j is a solution of the program:

$$\min_x \left\{ p^j.x : x \in L(y^j) \right\} \tag{3.12}$$

for all $j \in J$. Equivalently, a family of input sets $L(y)$ c-rationalizes the data if for all $j \in J$ and all $x \in L(y^j)$:

$$p^j.x^j \leq p^j.x. \tag{3.13}$$

Assume that the output set is one-dimensional ($n = 1$). The main difference with Varian's (1984) WACM is that here prices can be negative. This excludes the strong disposal (or positive monotonic) property of the input set. Following Varian (1984) we assume the family of input sets is nested by the following assumption:

$$\forall x \in L(y),\ 0 \leq v \leq y \quad \text{implies that} \quad x \in L(v). \tag{3.14}$$

In the following, we denote

$$I_j = \left\{ i \in [m] : p_i^j < 0 \right\}. \tag{3.15}$$

The key idea of the following result is that if an input requirement set $L(y)$ c-rationalizes the data, then it necessarily satisfies a minimal S-disposal assumption where

$$S = \bigcup_{j \in J} I_j. \tag{3.16}$$

This also means that $L(y)$ is S'-congested for all $S' \subsetneq S$.

Proposition 3.9 *The following conditions are equivalent:*

(a) *There exists a family of nested input sets $L(y)$ that c-rationalizes the data.*
(b) *If $y^k \geq y^j$, then $p^j.x^k \geq p^j.x^j$ for all $j, k \in J$.*
(c) *There exists a family of nontrivial closed, convex input sets that c-rationalizes the data and that satisfies a minimal S-disposal assumption, where $S = \bigcup_{j \in J} I_j$.*

An immediate consequence is that negative prices imply congestion of the technology. Obviously, if all observed prices are nonnegative, then we have $I_j = \varnothing$ for $j \in J$ and, because of $S = \varnothing$, we retrieve the Varian (1984) WACM result as a special case.

Notice that in principle it is possible to relax the convexity assumption (e.g., as in Briec et al. 2004). Obviously, the same remarks as the ones mentioned at the end of Sect. 3.1 apply.

4 Conclusions

Starting from a limited version of the widespread strong disposal assumption we define new technologies capable to model more general notions of congestion. The new S-disposal assumption relaxes standard disposability concepts (i.e., strong and ray disposability notions) by allowing a limited disposability with respect to a more general class of orders that generate joint restrictions on subsets of variables. In fact, the S-disposal assumption can be seen as an attempt to re-interpret the traditional strong disposal axiom as a “local” (in the sense of limited) instead of a global property (an issue already raised in Lau 1974). These new technologies lead to the formulation of a new duality result between the input directional distance function and the cost function with possibly negative prices. This duality result is considerably weaker than the results available in the current literature. Furthermore, it turns out that the S-disposal assumption allows modeling more general forms of congestion as defined in Färe and Svensson (1980) compared to the ray disposal hypothesis. Indeed, apart from monotone output limitational congestion that can also be represented by ray disposable input sets, technologies with S-disposal of inputs can also model output prohibitive congestion, which cannot be represented by ray disposable input sets.

One main limitation is that we focus on congestion in the input space solely. Therefore, we concentrate on the input directional distance functions and its dual relation with the cost function to characterize congestion. Generalizations to congestion phenomena in the outputs space or to the input and output space are relatively straightforward, but are deferred to a later contribution. Note that the use of the directional distance function allows for an easy extension of our proposals to define congestion in the output space or in the input–output space. Another limitation is that we ignore the consequences of relaxing the strong disposal assumption for general equilibrium theory. Indeed, it is rather well-known that the free disposal assumption cannot be dropped or relaxed in any way without risking that equilibria may fail to exist in non-convex (e.g., increasing returns to scale) economies (however, Chavas and Briec 2012 recently show that the use of a similar directional distance function framework in fact facilitates the analysis).

Straightforward extensions of this contribution are the development of empirical production models capable to test the different disposability assumptions (strong, weak, and S-disposal). Given the above generalization of the WACM result in Varian (1984), especially the definition of nonparametric technologies allowing for nonoptimizing behavior of a subset of observations should be relatively easy. These new technology specifications allow testing whether traditional assumptions like strong and weak disposal of inputs can be maintained against the more general S-disposal assumption. This testing framework could then extend the battery of tests verifying various combinations of strong and weak input disposability in both inputs and outputs in the literature. Furthermore, it could be interesting to empirically assess what difference the S-disposal axiom makes compared to the weak disposal hypothesis in terms of the shadow prices for bad outputs when explicitly modeling trade-offs between good and bad outputs (e.g., along the lines of Coggins and Swinton 1996).

References

Arrow K, Barankin E, Blackwell D (1953) Admissible points of convex sets. In: Kuhn H, Tucker A (eds) Contributions to the theory of games, vol II. Princeton University Press, Princeton, pp 87–91
Barnett W (2002) Tastes and technology: curvature is not sufficient for regularity. J Econometr 108(1):199–202
Blow L, Browning M, Crawford I (2008) Revealed preference analysis of characteristics models. Rev Econ Stud 75(2):371–389
Blundell R (2005) How revealing is revealed preference? J Eur Econ Assoc 3(2–3):211–235
Briec W, Kerstens K, Vanden Eeckaut P (2004) Non-convex technologies and cost functions: definitions, duality and nonparametric tests of convexity. J Econ 81(2):155–192
Chambers R, Chung Y, Färe R (1996) Benefit and distance functions. J Econ Theory 70(2):407–419
Chavas J-P, Briec W (2012) On economic efficiency under non-convexity. Econ Theory 50(3):671–701
Chavas J-P, Kim K (2007) Measurement and sources of economies of scope: a primal approach. J Inst Theoret Econ 163(3):411–427
Coggins J, Swinton J (1996) The price of pollution: a dual approach to valuing SO_2 allowances. J Environ Econ Manag 30(1):58–72
Dasgupta P, Mähler K-G (2003) The economics of non-convex ecosystems: introduction. Environ Res Econ 26(4):499–525
Duranton G, Turner M (2011) The fundamental law of road congestion: evidence from US cities. Am Econ Rev 101(6):2616–2652
Färe R, Grosskopf S (1983) Measuring congestion in production. J Econ 43(3):257–271
Färe R, Grosskopf S (2000) Decomposing technical efficiency with care. Manag Sci 46(1):167–168
Färe R, Jansson L (1976) Joint inputs and the law of diminishing returns. Zeitschrift für Nationalökonomie 36(3–4):407–416
Färe R, Svensson L (1980) Congestion of production factors. Econometrica 48(7):1745–1753
Ferguson C (1969) The neoclassical theory of production and distribution. Cambridge University Press, Cambridge
First Z, Hackman S, Passy U (1993) Efficiency estimation and duality theory for nonconvex technologies. J Math Econ 22(3):295–307
Fuss M, McFadden D, Mundlak Y (1978) A survey of functional forms in the economic analysis of production. In: Fuss M, McFadden D (eds) Production economics: a dual approach to theory and applications, vol 1. North-Holland, Amsterdam, pp 219–268
Hackman S (2008) Production economics: integrating the microeconomic and engineering perspectives. Springer, Berlin
Henderson D, Russell R (2005) Human capital and convergence: a production-frontier approach. Int Econ Rev 46(4):1167–1205
Jacobsen S (1970) Production correspondences. Econometrica 38(5):754–771
Kuosmanen T (2003) Duality theory of non-convex technologies. J Prod Anal 20(3):273–304
Kuosmanen T (2005) Weak disposability in nonparametric production analysis with undesirable outputs. Am J Agric Econ 87(4):1077–1082
Lau L (1974) Comments. In: Intrilligator M, Kendrick D (eds) Frontiers of quantitative economics, vol II. North-Holland, Amsterdam, pp 176–199
Luenberger D (1992) Benefit function and duality. J Math Econ 21(5):461–481
Luenberger D (1995) Microeconomic theory. McGraw-Hill, Boston
McDonald J (1996) A problem with the decomposition of technical inefficiency into scale and congestion components. Manag Sci 42(3):473–474
McFadden D (1978) Cost, revenue and profit functions. In: Fuss M, McFadden D (eds) Production economics: a dual approach to theory and applications. North-Holland, Amsterdam, pp 3–109
Murty S, Russell RR, Levkoff SB (2012) On modeling pollution-generating technologies. J Environ Econ Manag 64(1):117–135
Paris Q (2008) Law of the minimum. In: Chesworth W (ed) Encyclopedia of soil science. Springer, New York, pp 431–437
Scarf H (1986) Testing for optimality in the absence of convexity. In: Heller W, Starr R, Starrett S (eds) Social choice and public decision making: essays in honor of Kenneth J. Arrow, vol I. Cambridge University Press, Cambridge, pp 117–134

Shephard R (1974) Indirect production functions. Verlag Anton Hain, Meisenheim am Glam
Varian H (1984) The nonparametric approach to production analysis. Econometrica 52(3):579–597
Zhengfei G, Oude Lansink A (2003) Input disposability and efficiency in dutch arable farming. J Agric Econ 54(3):467–478